

Theory of unitary Bose gases

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We develop an analytical approach for the description of an atomic Bose gas at unitarity. By focusing in first instance on the evaluation of the single-particle density matrix, we derive several universal properties of the unitary Bose gas, such as the chemical potential, the contact, the speed of sound, the condensate density and the effective interatomic interaction. The theory is also generalized to describe Bose gases with a finite scattering length and then reduces to the Bogoliubov theory in the weak-coupling limit.

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Introduction. — Although strongly interacting systems are known to be notoriously difficult to describe from first principles, remarkable progress in our understanding has been made in recent years. For example, the universal nature of fermionic many-body systems with resonant two-body interactions has been successfully studied experimentally [1]. This has been achieved by utilizing the high degree of control available in ultracold atomic gas experiments, which allow the investigation of many-body systems from the weakly to the strongly interacting case, by using a magnetic-field-tunable Feshbach resonance [1–3]. The remarkable property of such resonant systems, which have an infinite scattering length and are therefore said to be at unitarity, is that at zero temperature there is no other length scale than the average interatomic distance that is set by the particle density n . As a result all thermodynamic quantities, when appropriately scaled, can then be expressed in terms of a set of universal numbers. One of the most crucial quantities of the Fermi gas at unitarity is the chemical potential

$$\mu = (1 + \beta)\epsilon_F, \quad (1)$$

which is given by an universal constant times the Fermi energy $\epsilon_F = \hbar^2 k_F^2 / 2m$, where $k_F = (6\pi^2 n / 2s + 1)^{1/3}$ is the Fermi momentum and $s = 1/2$ due to the hyperfine degrees of freedom. The universal constant β can be interpreted as describing the deviation from the ideal gas result due to interactions and was found to be $\beta \simeq -0.63$ experimentally as well as theoretically [4, 5].

Not only fermionic many-body systems are studied with ultracold atomic gases, but also its bosonic counterpart can be experimentally realized. In recent years, there has therefore been an increasing interest in the experimental study of the strongly interacting Bose gas at low temperatures. It is expected on dimensional grounds that the Bose gas at unitarity, if stable, has similar universal properties as that of the unitary Fermi gas. For instance Eq. (1) is expected to hold also but with $s = 0$ and a different value of β due to the different statistics of the atoms. In contrast to fermionic cold atomic gases, the experimental study of bosons with increasing inter-

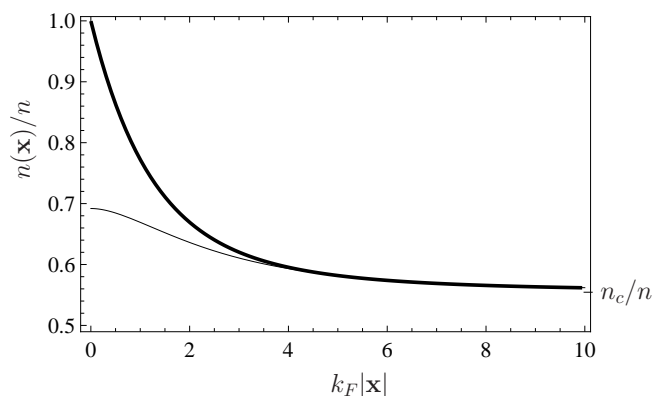


Figure 1. The universal one-particle density matrix $n(\mathbf{x})/n$ as a function $k_F|\mathbf{x}|$, where the thin line is the contribution of the condensate and its phase fluctuations as mentioned in the text. From the difference of the graphs the contribution coming from all other fluctuations can be inferred. The condensate density n_c/n is indicated on the right.

action strength is complicated by the loss of atoms, as a consequence of a strong increase in the rate of inelastic three-body recombination processes caused by the absence of the Pauli principle and the existence of Efimov trimers. As a result, up to now only a lower bound of $\beta > -0.56$ was determined experimentally [6]. These inelastic three-body processes result in the formation of molecules, which shows that the actual ground state of these gases is a Bose-Einstein condensate of molecules. Nevertheless, it may still be experimentally possible to create the meta-stable state of a Bose-Einstein condensate of atoms at large scattering lengths for a sufficiently long time. In view of this possibility, there has been considerable theoretical interest in the unitary Bose gas, but recent theoretical results strongly vary from $\beta \simeq -0.34$ using renormalization-group techniques [7], $\beta \simeq 1.93$ and $\beta \simeq -0.2$ from variational (Jastrow) analyses [8, 9], to the complete instability of the Bose gas at unitarity using a Noziers-Schmitt-Rink-like approach [10]. This clearly indicates that a systematic approach for the meta-stable atomic Bose gas at unitarity is still lacking.

In this Letter we present an analytical approach to the Bose gas at unitarity that can be improved systematically. It is based on the realization that fluctuations in the phase of the Bose-Einstein condensate dominate the long-wavelength behavior of the system [11–13]. Going beyond the Bogoliubov theory, which is required at unitarity, these gapless phase fluctuations lead to infrared divergencies, which make it increasingly difficult to apply resummation procedures to find, for example, the effective interaction at unitarity. Our approach circumvents these troublesome divergencies by exactly incorporating the phase fluctuations, which will be confirmed by reproducing the exact form of the single-particle propagator in the long-wavelength limit as derived by Nepomnyashchii and Nepomnyashchii [14, 15]. The approach thus consists of isolating the phase fluctuations of the condensate, which is reminiscent of bosonization for fermions. In addition, the theory is first renormalized due to all other fluctuations using the renormalization group. One of the outcomes of our theory, the universal one-particle density matrix, is shown in Fig. 1. The thin line indicates the contributions to the density matrix from the condensate and its phase fluctuations, from which it is seen that these fluctuations are vital to the correct description of the Bose gas at unitarity.

Renormalized Bosonization. — The Euclidean action of a Bose gas with a point interaction is given by $S[\phi^*, \phi] = \int d\tau d\mathbf{x} \mathcal{L}$, where the Lagrangian density is

$$\mathcal{L} = \phi^* \left[\hbar \partial_\tau - \frac{\hbar^2 \nabla^2}{2m} - \mu \right] \phi + \frac{1}{2} T^{2B} |\phi|^4, \quad (2)$$

$\phi(\mathbf{x}, \tau)$ is the bosonic field, and the strength of the interaction is characterized by $T^{2B} = 4\pi a \hbar^2 / m$ with a the s -wave scattering length. To describe the Bose-Einstein condensate at zero temperature, we expand the field as

$$\phi(\mathbf{x}, \tau) = \sqrt{n_0} \exp[i\theta(\mathbf{x}, \tau)] + \phi'(\mathbf{x}, \tau), \quad (3)$$

where n_0 should be viewed as the quasicondensate density [3] and is not the density of atoms in the condensate n_c , as illustrated by Fig. 1. Roughly speaking, the first term of the expansion describes the low-energy modes of the field, as shown in Fig. 2, and includes the phase fluctuations. The field $\phi'(\mathbf{x}, \tau)$ describes the high-energy modes and is defined such that it does not contain phase fluctuations and is thus orthogonal to the first term in Eq. (3). By inserting the expansion into Eq. (2), the action $S[n_0, \theta, \phi'^*, \phi']$ is obtained. There are several properties of this action worth mentioning.

First of all, the dominant low-energy physics is known to be due to the phase fluctuations. It can be shown that the exact form of the phase-fluctuation action $S[\theta]$ in the long-wavelength limit is

$$\frac{1}{2} \sum_{\mathbf{k}, n} \theta^*(\mathbf{k}, \omega_n) \left[\frac{mc^2/n}{(\hbar\omega_n)^2 + 2mc^2\epsilon_{\mathbf{k}}} \right]^{-1} \theta(\mathbf{k}, \omega_n), \quad (4)$$



Figure 2. Schematic representation of the expansion of the field ϕ in terms of the condensate and its phase fluctuations and the non-phase fluctuations ϕ' , c.f. Eq. (3).

where $\omega_n = 2n\pi/\hbar\beta$ is the Matsubara frequency, $\beta^{-1} = k_B T$ is the temperature, \mathbf{k} is the wave vector, $\epsilon_{\mathbf{k}} = \hbar^2 \mathbf{k}^2 / 2m$ is the atomic dispersion, and c is the speed of sound. In other words, Eq. (4) can formally be obtained from the full action $S[n_0, \theta, \phi'^*, \phi']$ by integrating out all non-phase fluctuations ϕ' and the fluctuations in n_0 .

Second, the accuracy of the action $S[n_0, \theta, \phi'^*, \phi']$ can be improved systematically by incorporating the ϕ' fluctuations into a renormalization of the action. However, it turns out to be more convenient to carry out this renormalization at the level of $S[\phi^*, \phi]$, Eq. (2), and then apply the expansion of the field, as in Eq. (3). The exact renormalization-group flow equation for the action $S[\phi^*, \phi]$ is

$$\frac{dS}{d\Lambda} = \frac{\hbar}{2} \text{Tr} \delta_\Lambda \ln \left[-\mathbf{G}'^{-1} + \frac{1}{\hbar} \frac{\delta^2 S_{\text{int}}}{\delta \Phi \delta \Phi^*} \right].$$

Here $S[\phi^*, \phi; \Lambda]$ is the effective action obtained by integrating out all non-phase fluctuations above the momentum $\hbar\Lambda$, \mathbf{G}' is the matrix propagator of the non-phase fluctuations, S_{int} is the non-gaussian part of the effective action, the trace is over space, imaginary time and Nambu space $\Phi(\mathbf{k}, \omega_n) = [\phi'(\mathbf{k}, \omega_n), \phi'^*(-\mathbf{k}, -\omega_n)]^T$, and $\delta_\Lambda = \delta(k - \Lambda)$. Although there are no small parameters in the theory of unitary Bose gases, the renormalization group can distinguish between the relevance of the various coupling constants based on their scaling dimension under renormalization. As the effective interaction evaluated at zero momentum and zero frequency is expected to be a crucial variable, which however also induces a flow of the chemical potential, let us here restrict our attention to these parameters. The running of the chemical potential and effective interaction are in general found to be given by $\Lambda d\mu/d\Lambda = \beta_\mu(\mu, g)$ and $\Lambda dg/d\Lambda = \beta_g(\mu, g)$. By solving these equations the renormalized action $S[\phi^*, \phi; \Lambda]$ is found. Then, after inserting the expansion of the field, the renormalized action $S[n_0, \theta, \phi'^*, \phi'; \Lambda]$ is obtained. This action defines the propagator of the non-phase fluctuations in terms of the effective interaction.

Lastly, to actually perform the above-mentioned renormalization, the propagator of the non-phase fluctuations ϕ' is needed. It is obtained by realizing that these fluctuations are determined from the action $S[n_0, \theta, \phi'^*, \phi'; \Lambda]$ with a non-fluctuating phase. The quadratic part of the action is then written as $S[\phi'^*, \phi'] = \frac{1}{2} \sum_{\mathbf{k}, n} \Phi^\dagger(\mathbf{k}, \omega_n) [-\hbar \mathbf{G}'^{-1}(\mathbf{k}, \omega_n)] \Phi(\mathbf{k}, \omega_n)$, where the

components of the symmetric 2×2 Green's function of the non-phase fluctuations are given by

$$\begin{aligned} -G'_{11}(\mathbf{k}, \omega_n) &= \hbar \frac{i\hbar\omega_n + \epsilon_{\mathbf{k}} + n_0 g}{(\hbar\omega_n)^2 + (\hbar\omega_{\mathbf{k}})^2} - n_0 \langle \theta\theta^* \rangle(\mathbf{k}, \omega_n), \\ -G'_{12}(\mathbf{k}, \omega_n) &= \hbar \frac{-n_0 g}{(\hbar\omega_n)^2 + (\hbar\omega_{\mathbf{k}})^2} + n_0 \langle \theta\theta^* \rangle(\mathbf{k}, \omega_n), \end{aligned}$$

with the Bogoliubov dispersion $(\hbar\omega_{\mathbf{k}})^2 = \epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2n_0 g)$.

The first term in the propagator is the standard Bogoliubov propagator, which also includes the effect of phase fluctuations. The second term precisely removes these effects, which can for instance be identified by their proportionality to n_0 in the numerator. For consistency, we define $mc^2 \equiv n_0 g$ in accordance with the Bogoliubov dispersion. Notice, however, that the residues of the exact propagator of the phase fluctuations incorporate the renormalization of the superfluid density from n_0 to n , which is not present in the Bogoliubov propagator.

After the subtraction of the phase fluctuations, the propagator of the non-phase fluctuations is given by

$$\hbar^{-1} \langle \phi'(\mathbf{k}, \omega_n) \phi'^*(\mathbf{k}, \omega_n) \rangle = \frac{i\hbar\omega_n + \epsilon_{\mathbf{k}}}{(\hbar\omega_n)^2 + (\hbar\omega_{\mathbf{k}})^2}, \quad (5)$$

and the anomalous averages vanish, i.e., $\langle \phi' \phi' \rangle = \langle \phi'^* \phi'^* \rangle = 0$. The vanishing of the anomalous averages means that the Green's function \mathbf{G}' is diagonal and this greatly simplifies the renormalization procedure of the interaction, which justifies our preference for solving the renormalization-group equation for $S[\phi^*, \phi]$.

Before we turn to the determination of the effective interaction, we first show that our approach reproduces the exact propagator in the static long-wavelength limit derived by Nepomnyashchii and Nepomnyashchii, as mentioned in the introduction. This is achieved using the one-particle correlation function $\langle \phi(\mathbf{x}, \tau) \phi^*(\mathbf{0}, 0) \rangle$, which equals $n_0 \langle \exp[i(\theta(\mathbf{x}, \tau) - \theta(\mathbf{0}, 0))] \rangle + \langle \phi'(\mathbf{x}, \tau) \phi'^*(\mathbf{0}, 0) \rangle$, and taking the Fourier transform. By expanding the exponential, we find that the dominant long-wavelength behavior is due only to the first two terms in the expansion, where the first term is simply the exact phase-fluctuation propagator in Eq. (4) while the second term comes from a convolution of two propagators, namely

$$\begin{aligned} \hbar^{-1} \langle \phi(\mathbf{k}, \omega_n) \phi^*(\mathbf{k}, \omega_n) \rangle &\simeq \frac{\frac{n_c}{n} mc^2}{(\hbar\omega_n)^2 + 2mc^2 \epsilon_{\mathbf{k}}} \\ &- \frac{3\sqrt{mc^2}}{32\sqrt{2}\epsilon_F^{3/2}} \frac{n_c}{n} \log \left[\frac{(\hbar\omega_n)^2 + 2mc^2 \epsilon_{\mathbf{k}}}{(8mc^2)^2} \right] + n_c \beta V \delta_{\mathbf{k}, \mathbf{0}} \delta_{n, 0}, \end{aligned}$$

which is the form of the exact propagator [14, 15] with n_c the condensate density defined later in Eq. (6) and V is the volume. The exact anomalous propagator is also reproduced, and is given by $n_0 \langle \exp[i(\theta(\mathbf{x}, \tau) + \theta(\mathbf{0}, 0))] \rangle$. In particular, this leads to the counter-intuitive conclusion that the anomalous self-energy vanishes at zero momentum and frequency [15].

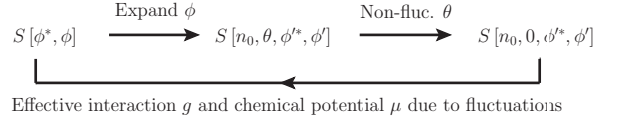


Figure 3. Schematic representation of the renormalization procedure that shows the self-consistent nature of our theory, as described in the text.

To summarize and as illustrated in Fig. 3, the action $S[\phi^*, \phi]$ of the Bose gas can be systematically renormalized by the non-phase fluctuations ϕ' using the renormalization-group flow equation, giving in particular rise to an effective coupling g and a renormalized chemical potential μ . The propagators of the non-phase fluctuations are determined self-consistently after expansion of the field. The theory includes the exact propagator of phase fluctuations and can reproduce the exact propagator $\langle \phi\phi^* \rangle$ in the long-wavelength limit.

Universal Results. — Now that our theoretical framework has been established it is possible to determine, for example, the condensate density, the effective interaction and the chemical potential at unitarity. The condensate density can be calculated from the off-diagonal long-range order, namely $n_c \equiv \lim_{|\mathbf{x}| \rightarrow \infty} \langle \phi(\mathbf{x}, 0) \phi^*(\mathbf{0}, 0) \rangle$,

$$n_c = n_0 \exp \left[\frac{3}{4} (2\sqrt{2} - \pi) \left(\frac{n_0 g}{\frac{\hbar^2}{2m} (6\pi^2 n_0)^{2/3}} \right)^{3/2} \right], \quad (6)$$

where it must be noted that an ultra-violet subtraction is needed in order to calculate the expectation value of the phase fluctuations using Eq. (4) with $mc^2 = n_0 g$ [3]. This ultra-violet subtraction removes the ultra-violet divergences associated with a point interaction, and is a result of the renormalization of the bare coupling to T^{2B} . In order to determine the condensate density at unitarity, the quasicondensate density n_0 needs to be eliminated in favor of the total density $n = \langle \phi(\mathbf{x}, \tau) \phi^*(\mathbf{x}, \tau) \rangle$ using

$$n = n_0 + \frac{(8\sqrt{2} - 3\pi)}{24\pi^2} \left(\frac{2m}{\hbar^2} [n_0 g] \right)^{3/2}, \quad (7)$$

where the second term is the contribution from the high-energy fluctuations $n' = \langle \phi'(\mathbf{x}, \tau) \phi'^*(\mathbf{x}, \tau) \rangle$, see Eq. (5). As required, exactly the same ultra-violet subtraction was used for the high-energy fluctuations as in Eq. (6).

To proceed, we must now determine the effective interaction g in some approximation. Taking only the renormalization of the coupling constant and the chemical potential into account, which turns out to be very accurate for the unitary Fermi gas [16], the beta functions are given by

$$\beta_\mu = -2g \frac{4\pi\Lambda^3}{(2\pi)^3} \left[|v_\Lambda|^2 + \frac{n_0 g}{2\epsilon_\Lambda + 2n_0 g} \right], \quad (8)$$

$$\beta_g = g^2 \frac{4\pi\Lambda^3}{(2\pi)^3} \left[\frac{|u_\Lambda|^4 + |v_\Lambda|^4 - 8|u_\Lambda|^2 |v_\Lambda|^2}{2\hbar\omega_\Lambda} - \frac{1}{2\epsilon_\Lambda} \right],$$

where the Bogoliubov dispersion $\hbar\omega_{\mathbf{k}}$ and the coherence factors $|u_{\mathbf{k}}|^2 = |v_{\mathbf{k}}|^2 + 1 = (\hbar\omega_{\mathbf{k}} + \epsilon_{\mathbf{k}})/2\hbar\omega_{\mathbf{k}}$ are evaluated at Λ . The effective interaction is obtained by integrating its differential equation using the boundary condition $g(\Lambda = \infty) = T^{2B}$, where it must be noted that the effective interaction inside the Bogoliubov dispersion is the fully renormalized value $g(\Lambda = 0)$ which, as previously explained, is determined self-consistently. Ultimately, we obtain

$$\frac{1}{g} = \frac{1}{T^{2B}} + \frac{1}{4\sqrt{2}\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{n_0 g}. \quad (9)$$

Note that this equation can also be obtained directly as the result of a resummation of an infinite number of the diagrams shown in Fig. 4. The equation also shows the highly non-perturbative nature of the renormalization group.

In the unitarity limit, $T^{2B} \rightarrow \infty$, the effective interaction and the condensate density in terms of the total density are found by solving Eqs. (6-9) to be

$$\frac{ng}{\epsilon_F} = \frac{2}{3^{2/3}} (1 + \lambda)^{1/3} \simeq 1.09, \quad \frac{n'}{n} = \frac{\lambda}{1 + \lambda} \simeq 0.31,$$

$$\frac{n_c}{n} = \frac{1}{1 + \lambda} \exp \left(\frac{2\sqrt{2} - \pi}{\sqrt{2}} \right) \simeq 0.55,$$

where $\lambda = n'/n_0 = (8\sqrt{2} - 3\pi)/3\sqrt{2} \simeq 0.45$. The depletion from the condensate is given by $1 - n_c/n \simeq 0.45$, which clearly differs from the density of particles contributing to the non-phase fluctuating modes n' by phase-fluctuation contributions.

The universal one-particle density matrix $n(\mathbf{x}) = \langle \phi(\mathbf{x}, 0) \phi^*(\mathbf{0}, 0) \rangle$ is shown in Fig. 1 and can be seen to reduce in the large-distance limit to the condensate density. Also the contribution due to the phase fluctuations is indicated, which terminates at n_0/n at equal position. Another interesting property at unitarity is called the contact C and is related to the short-wavelength behavior of the one-particle density matrix, namely $n(\mathbf{k}) \simeq C/\mathbf{k}^4$



Figure 4. The Feynman diagrams for the fluctuations ϕ' that contribute to the beta function β_g and can be viewed as the building blocks for the ladder and bubble sums included in the Bethe-Salpeter equation for the effective interaction g .

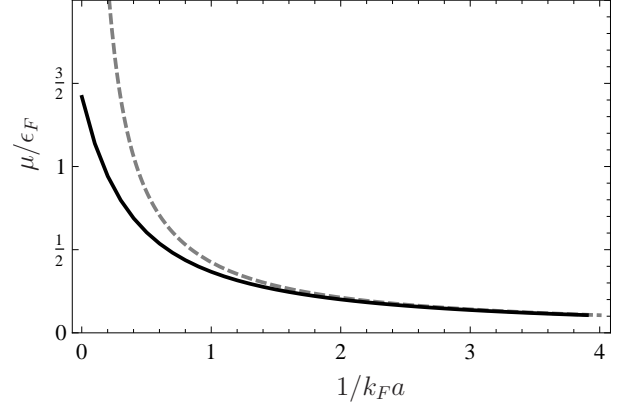


Figure 5. The chemical potential as a function of scattering length. The gray dashed line is the Bogoliubov chemical potential [3].

[17, 18]. The contact is

$$\frac{C}{k_F^4} = \left(\frac{n_0 g}{2\epsilon_F} \right)^2 = \frac{1}{3^{4/3}} \frac{1}{(1 + \lambda)^{4/3}} \simeq 0.14.$$

The change in the chemical potential follows from integrating Eq. (8) and is given by $\Delta\mu = 2n'g$. According to the exact Hugenholtz-Pines theorem the chemical potential in our theory is then given by $\mu = n_0 g + \Delta\mu$, such that the universal chemical potential is

$$\frac{\mu}{\epsilon_F} = \frac{n_0 g + 2n'g}{\epsilon_F} = \frac{2}{3^{2/3}} \frac{1 + 2\lambda}{(1 + \lambda)^{2/3}} \simeq 1.42,$$

which results in the universal constant $\beta \simeq 0.42$. Furthermore, the speed of sound at unitarity is given by

$$\frac{mc^2}{\epsilon_F} = \frac{n_0 g}{\epsilon_F} = \frac{1}{1 + 2\lambda} \frac{\mu}{\epsilon_F} \simeq 0.53 \frac{\mu}{\epsilon_F} \simeq 0.75.$$

The expected value for the speed of sound at unitarity in terms of the chemical potential is $mc^2 = n(d\mu/dn) = 2\mu/3 \simeq 0.66\mu$, which is remarkably close to our result and shows the accuracy of the simplest first approximation that we have presented here.

Conclusions. — We have constructed an approach to describe Bose gases at unitarity which can be improved systematically by renormalization-group methods. As an outlook, all quantities can be found not only at unitarity, but it is possible using Eq. (9) to find all these quantities from weak to infinitely strong coupling, as is shown for the chemical potential in Fig. 5. The generalization of the theory to frequency-dependent interactions and also non-zero temperature is straightforward and will be elaborated on in a future publication. Furthermore, we expect that the approach can be applied to other systems with a broken continuous symmetry, where similar infrared divergencies occur as a consequence of the Goldstone modes. We hope that our results stimulate fur-

ther experimental developments toward unitarity-limited Bose gases in the near future.

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